

# Constrained Systems in a Coarse-Grained Scenario

Cresus F. L. Godinho<sup>a,\*</sup>, Jose Weberszpil<sup>b,†</sup> and J. A. Helayël Neto<sup>c,‡</sup>

<sup>a</sup>*Grupo de Física Teórica, Departamento de Física,  
Universidade Federal Rural do Rio de Janeiro,  
BR 465-07, 23890-971, Seropédica, RJ, Brazil*

<sup>b</sup>*Universidade Federal Rural do Rio de Janeiro, UFRRJ-IM/DTL,  
Av. Governador Roberto Silveira s/n ,  
Nova Iguaçu, RJ, Brazil*

<sup>c</sup>*Centro Brasileiro de Pesquisas Físicas (CBPF),  
Rua Dr. Xavier Sigaud 150, Urca,  
22290-180, Rio de Janeiro, Brazil  
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A number of new approaches based on fractional calculus have been presented and discussed in the literature over the past years. Their purpose is to find out different perspectives and describe particular phenomena in connection with field theory and gravity at a more fundamental level. On the other hand, procedures related to canonical quantization are really essential in field theory. More recently, the investigation of a wide category of new classical field-theoretic models and their respective quantized counterparts have been pursued which yield a very rich scenario that enables us to connect different areas of physics. For instance, it is not fairly well-known how to deal with dissipative or nonlinear systems; there are many paths of investigation in the literature; none of them presents a systematic and general procedure to tackle the problem. However, it is widely accepted today that the fractional formalism may be viewed as a powerful alternative to study dissipative systems. In our present contribution, we propose a pragmatic and detailed way of addressing the question. Adopting a particular approach to fractional calculus, this paper sets out to build up a consistent extension of the Faddeev-Jackiw (or Symplectic) algorithm to carry out the quantization procedure of nonconservative models in the standard canonical way. In our treatment, we shall choose the so-called Modified Riemann Liouville (MRL) approach, where the chain rule is as workable as it is in the standard differential calculus. We believe that, by adopting the extended version of Fractional Symplectic Quantization procedure, it shall be possible to carry out a deeper analysis of gauge theories implemented in a coarse-grained scenario.

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## I. INTRODUCTION

A powerful tool idealized in the sixties, motivating a large production of papers concerning constrained systems, the Dirac brackets (DB) [1] were an unmodified common point in the literature of the subject. The main motivation of many works was to convert second-class systems into first-class ones. The main objective was to obtain gauge theories (first-class systems), the holy grail for the Standard Model. Although not so popular as before, the analysis of constrained systems still concentrates a great deal of attention in the literature [2].

The Symplectic Method [3] is a geometrical way of approaching canonical quantization of constrained systems. One of its main ingredients is the symplectic tensor, that plays the role of a metric in the symplectic manifold. Faddeev and Jackiw originally proposed a very interesting method which is centrally based on the attainment

of the inverse of the symplectic tensor. Its elements are immediately associated with the Dirac brackets and, consequently, it becomes possible to obtain the quantum commutators by means of the usual rule; of course, if no problems with the ordering of operators is present

$$f^{-1} \rightarrow \{ , \}^* \rightarrow \frac{1}{i\hbar} [ , ] . \quad (1)$$

The difficulty to obtain the symplectic tensor is proportional to the existence of constraints, as subsequently shown [4, 5]. It is important to highlight here that the constraints usually defined in Dirac's approach do not necessarily appear in the Symplectic Method. The constrained systems considered in both scenarios have constraints usually referred to as true constraints.

It is well-known that several physical systems may be studied in a Lagrangian approach; their coordinates are usually embedded in a phase space. Some of those Lagrangian systems can be written in a form where first-order time derivatives are present among the fields. The so-called kinetic part is written in terms of first-order time derivatives which constitute 1-forms whose exterior derivatives appear in the equations of motion. The resulting 2-form, the symplectic tensor, if singular, proba-

\* crgodinho@ufrj.br

† josewebe@ufrj.br

‡ helayel@cbpf.br

bly signals a constrained system [3]. If the system is not constrained, the inverse of the symplectic tensor usually exists and provides the fundamental Poisson brackets.

Constrained systems can be usually solved by obtaining the inverse form of the symplectic tensor. Faddeev and Jackiw applied the Darboux theorem to work with canonical and non-canonical sectors separately [5]. It is still possible to show that the Faddeev-Jackiw and the Dirac approaches are completely equivalent whenever constrained systems are under consideration [6]. However, it was shown that, if second-class constraints are present, the equivalence between the Dirac and Symplectic approaches fails [7].

On the other hand, there is a number of problems in considering classical systems besides the ones that involve the quantization of second-class systems, as we have just mentioned above. These problems encompass the so-called nonconservative systems. The peculiarity about them is that the great majority of actual classical systems is nonconservative but, in spite of that, the most advanced formalisms of classical mechanics deal only with conservative systems [8].

Dissipation, for example, is present even at the microscopic level. There is dissipation in every non-equilibrium or fluctuating process, including dissipative tunneling [9] and electromagnetic cavity radiation [10], for instance.

One way to suitably treat nonconservative systems is through Fractional Calculus, FC, since it can be shown that, for example, a friction force has its form stemming from a Lagrangian that contains a term proportional to the fractional derivative, which may be a derivative of any non-integer order [8].

Field theory aspects of non-linear dynamics are today an important subject of study in different physical and mathematical sub-areas, but the real success and a radically new understanding of non-linear processes has occurred over the past 40 years. This understanding was inspired by the discovery and insight of chaotic dynamics, where the randomness of some physical processes are considered; more precisely, when particle trajectories are indistinguishable for random process [11].

Fractional Calculus is one of the generalizations of the classical calculus. It has been used in several fields of science. FC provides a redefinition of mathematical tools and it seems very useful to deal with anomalous and frictional systems. In particular, we can cite the continuous time random walk scheme as a physical counterpart example, where, within the fractional approach, it is possible to include external fields in a straightforward manner. Also, the consideration of transport in the phase space spanned by both position and velocity coordinates is possible within the same approach. Moreover, the calculation of boundary value problems may be driven to a form analogous to the procedure adopted to treat the corresponding standard differential equations. [12–17]. Other important applications may be found by investigating response functions, for which many studies have been reported on the phenomenon of non-exponential,

power-law relaxation which is typically observed in complex systems, such as dielectric and ferroelectric systems. Moreover, describing dynamical processes in disordered or complex systems, such as relaxation or dielectric behavior in polymers or photo bleaching recovery in biologic membranes, has proven to be an extraordinarily successful tool. The main feature of such systems is a strong (in general, randomic) interaction between their components in the passage to a state of equilibrium. Some authors have proposed fractional relaxation models to describe filled polymer networks and investigate the dependence of a number of parameters on the filler content [18, 19]. The study of exactly solvable fractional models of linear viscoelastic behavior is another successful field of application. In recent years, both phenomenological and molecular-based theories for the study of viscoelastic materials have been proposed with integral or differential equations of fractional order. Some current models of viscoelasticity based on FC are usually derived from the Maxwell model by replacing the first-order derivative ( $d/dt$ ) by its fractional version ( $d^\alpha/dt^\alpha$ ) [20], where  $\alpha$  is not integer. Presently, areas such as field theory and gravitational models demand new conceptions and approaches which might allow us to understand new systems and could help in extending well-known results. Interesting problems may be related to the quantization of field theories for which new approaches have been proposed [21–23].

In this work, we shall use FC to analyze the well-established canonical quantization symplectic algorithm. The focus is to construct a generalized extension of that method to treat a broader number of mechanical systems with respect to the standard method. In this sense, we shall adopt the Modified Riemann Liouville (MRL) prescription for fractional derivative.

Since FC has not yet actually been explored enough in field theory, despite some really interesting recent contributions [24–27], we have tried to construct a self-sustained paper with content distributed as follows. In Section (II), we provide a short review on the symplectic algorithm along with its main equations and formulations. Next, in Section (III), we present some basic aspects of the MRL approach that fit for our purposes here. In Section (IV), we propose the so-called fractional extension for the symplectic scheme on two different scenarios: the first one, in its simpler form, the Lagrangian has the following structure:  $L = \eta^i f_{ij} {}_0D_t^\alpha \eta^j - V(\eta)$ . In the second, and more general scenario, the Lagrangian considered is  $L = a_i(\eta, \partial\eta) {}_0D_t^\alpha \eta^i - V(\eta)$ , where in the kinetic sector,  $(a_i D\eta)$ , we now have some general function of  $\eta, \partial\eta$ . In both cases, by using the fractional time derivative  ${}_0D_t^\alpha$  in the MRL formulation, we can reassess the symplectic scheme and obtain new dynamical consequences, as well as extensions for the symplectic matrix and Euler-Lagrange equations. In Section (V), we initially consider the Dirac-Bergmann formalism embedded in a coarse-grained scenario and show its connection with the symplectic approach. Finally, in Section (VI), our

Concluding Remarks are cast. We also present therein an example and comment on our results and possible new paths to be followed in forthcoming papers.

## II. A BRIEF REVIEW OF THE SYMPLECTIC SCHEME FOR CONSTRAINED SYSTEMS

Following the traditional prescription of the symplectic algorithm [3–5], the so-called first-order formalism, one starts by considering a Lagrangian which is first-order in time derivative, represented with its kinetic and potential sectors, respectively

$$L^{(0)} = p_k \dot{q}_k - V(p, q), \quad p_k, q_k, (k = 1, 2, \dots, 2n), \quad (2)$$

where we adopt the summation convention for repeated indices and we consider it hereafter. Introducing now  $4n$  bosonic phase-space variables labeled here as  $\eta$ , we can rewrite the Lagrangian in its canonical one-form

$$L^{(0)} = \frac{1}{2} \eta_i f^{(0)} \dot{\eta}^i - V(\eta), \quad (3)$$

where  $f^{(0)}$  is called symplectic matrix

$$f^{(0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (4)$$

The equations of motion can be obtained by taking the variation " $\delta L = 0$ ",

$$\dot{\eta}^i = (f^{(0)})_{ij}^{-1} \frac{\partial V}{\partial \eta^j}. \quad (5)$$

Adopting now a more extensive viewpoint, let us write the Lagrangian

$$L^{(0)} = a_i(\eta) \dot{\eta}^i - V^{(0)}(\eta), \quad (i = 1, 2, \dots, m) \quad (6)$$

It can be rewritten as

$$L^{(0)} dt = a_i(\eta) d\eta^i - V^{(0)}(\eta) dt, \quad (7)$$

where, from the one-form  $a_i(\eta) d\eta^i$ , we can obtain the two-form

$$\begin{aligned} f^{(0)} &= d[a_i(\eta) d\eta^i] \\ f^{(0)} &= \frac{1}{2} f_{ij}^{(0)} d\eta^i \wedge d\eta^j \\ (i, j) &= 1, \dots, m, \end{aligned} \quad (8)$$

and, consequently,

$$f_{ij}^{(0)} = \frac{\partial a_j}{\partial \eta^i} - \frac{\partial a_i}{\partial \eta^j}. \quad (9)$$

It is well-known that the symplectic matrix in such cases is an antisymmetric tensor. From the symplectic tensor, it is possible to define the Euler-Lagrange equations

$$\dot{\eta}^j = [f_{ij}^{(0)}]^{-1} \frac{\partial V^{(0)}}{\partial \eta^i}. \quad (10)$$

The main reason to approach the problem in this way is that the symplectic matrix has a very close relation to the Dirac brackets  $\{, \}^*$ ,

$$f_{ij}^{-1} = \{, \}^*. \quad (11)$$

A very important point should be pointed out when  $f_{ij}$  is singular. If this is the case, the symplectic tensor is not well-defined and the bracket structures are not yet accessible. This situation probably points to a constrained system and, as such, we must proceed in a non-standard way. First of all,  $f_{ij}$  is not invertible and, according to the symplectic algorithm, we must obtain the zero-modes  $\nu_n$  satisfying the following relation

$$f_{ij}^{(0)} \nu_n^{(0)} = 0; \quad (12)$$

combining it with (10), we obtain the very useful relation

$$f_{ij}^{(0)} \nu_n^i \dot{\eta}^j = \nu_n^i \frac{\partial V^{(0)}}{\partial \eta^i} = 0. \quad (13)$$

This may be a constraint, but not exactly in a Dirac sense. These constraints are usually introduced in the Lagrangian by means of Lagrangian multipliers which "deform" the kinetic sector of the Lagrangian. This can be done by taking the time-derivative of the constraint and making use of some Lagrange multiplier. These procedures will enlarge the configuration space of the theory, so that we can identify new vectors,

$$a_i^{(1)} = a_i^{(0)} + \lambda_m^{(0)} \partial_i \Omega_m^{(0)} \quad (14)$$

where  $\Omega_m^{(0)}$  are the constraints obtained from (13). However, it may also occur that we arrive at a point where we still obtain a singular matrix and the corresponding zero modes do not show us any new constraint. This is a strict case of gauge theory. In such a situation, it is necessary to impose some specific gauge condition.

## III. THE MRL APPROACH

It is well-known that several definitions of fractional derivative and fractional integral exist; for instance, Grunwald-Letnikov, Riemann-Liouville, Caputo, Weyl, Feller, Erdelyi-Kober and Riesz fractional derivatives as well as fractional Liouville operators, which have been popularized when fractional integration is performed in the framework of the dynamical systems under study [28]. Following this idea, let us consider an approach recently proposed which subtly modifies the usual Riemann-Liouville, known as MRL [29, 30]. The main reason is to by-pass the pitfalls of Riemann-Liouville and Caputo definitions, where the derivative of a constant is not zero and it is necessary to have a higher-order derivative to evaluate the lower-order derivative. Moreover, the chain

rule, when considered in such cases, shows itself an unpractical exercise; but, in the MRL framework, it has almost the same shape as in the usual calculus.

In this sense, the fractional derivative of order  $\alpha, \alpha < 0$  of a given function  $f(x)$  is given by,

$$\begin{aligned} f^\alpha(x) &= \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha-1} \Delta f d\xi, \quad \alpha < 0, \\ (f^{(\alpha-1)}(x))' &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha} \Delta f d\xi, \\ 0 &< \alpha < 1. \end{aligned} \quad (15)$$

where we have considered that

$$\Delta f = f(\xi) - f(0). \quad (16)$$

Following this approach, let us consider now two functions,  $f(x)$  and  $u(x)$ . Depending upon the nature of the functions, we can work with the following chain rules: if  $f(u)$  is  $\alpha$ -th differentiable and  $u(x)$  is differentiable w.r.t.  $x$  then

$$\frac{d^\alpha f(u(x))}{dx^\alpha} = \frac{d^\alpha f(u)}{du^\alpha} \left( \frac{du}{dx} \right)^\alpha, \quad (17)$$

if  $f(u)$  is differentiable w.r.t.  $u$ , but not differentiable w.r.t.  $x$  and  $u$  is  $\alpha$ -th differentiable w.r.t.  $x$

$$\frac{d^\alpha f(u(x))}{dx^\alpha} = \frac{df(u)}{du} \left( \frac{d^\alpha u}{dx^\alpha} \right). \quad (18)$$

We would like to remark here that we are not making any sort of criticism to other approaches to fractional derivative; we have to remember that, today, we can work with several different definitions. We really have a very rich set of options, but it seems that each definition possesses particular characteristics and, so, we must suitably apply them in some physical context. In our case, we believe that the MRL set-up is sufficient to our main purposes, mainly by virtue of its very simplified Leibniz and chain rules. On the top of that, we are able to deal with non-differentiable functions in a coarse-grained context.

#### IV. THE SYMPLECTIC SCHEME AND THE MRL CONTEXT

In this section, we present an overview of how to obtain the symplectic algorithm by adopting an alternative approach. While making no claims of originality, our point of view is very singular when compared with other procedures that appear in the literature. Then, let us consider the simplest case of a field theory, whose the action is given by

$$\begin{aligned} S &= \int d^4x [\pi_k(x)_0 D_t^\alpha \phi_k(x) - H(\pi, \phi)] \\ k &= 1, 2, \dots, 2n \end{aligned} \quad (19)$$

where, for our purposes, we now adopt  ${}_0D_t^\alpha$  as the time fractional derivative in MRL sense (15).

We rewrite the Lagrangian as

$$S = \int d^4x \left[ \frac{1}{2} \pi_k(x)_0 D_t^\alpha \phi_k(x) - \frac{1}{2} \phi_k(x)_0 D_t^\alpha \pi_k(x) - H \right], \quad (20)$$

Two points could be highlighted here: we have changed the notation of the time derivative operator to avoid confusing it with the main definition of (15); the fields are not yet being considered in a coarse-grained scenario. We are proposing only a fractional extension to the action, considering the fractional time derivative that acts on the fields and its possible unfolds. Introducing now  $4n$  new (bosonic) phase space variables:  $(\eta_i) = (p, q), i = 1, 2, \dots, 4n$ . It is possible to write the Lagrangian in a symplectic context,

$$S = \left[ \frac{1}{2} \eta^i f^{(0)} {}_0D_t^\alpha \eta_i - H(\eta) \right], \quad (21)$$

where, once more, we find the same symplectic matrix as the one given in (4). However, a subtle and expected difference occurs when we obtain the equations of motion by the action variation,  $\delta S = 0$ , implying a very reasonable relation

$${}_0D_t^\alpha \eta^i = [f^{(0)}]_{ij}^{-1} \frac{\partial H}{\partial \eta^j}. \quad (22)$$

We obtain a new result, representing a fractional extension for the Hamilton-Jacobi equations in the symplectic formulation. The consistency of the method imposes the natural limit if  $\alpha \rightarrow 0$ .

The result found above probably points to a memory effect, induced in a dynamical system when an evolution described through a fractional differential equation is considered. With the general definition (15), the fractional derivative accounts for such a memory effect due to its dependence on many time moments. Its applicability could be justified for systems with dissipation effects where friction forces would be present. A very close result was obtained by [31] by adopting a Riemann-Liouville derivative.

Considering the fact that there are systems for which it is necessary to consider a more detailed and deep Lagrangian description, it is interesting to think in a more general sense. In this way, actions like

$$S = \int [a_i(\eta)_0 D_t^\alpha \eta^i - V(\eta)] d^4x, \quad i = 1, \dots, 4n \quad (23)$$

will be very helpful.

It is well-known that Lagrangians with higher orders in time derivative can be put in the first-order formulation too, considering some specific and well-tested field redefinitions. In order to be more explicit in our considerations about the canonical structure, let us take once more the variation of the action

$$\delta S = 0, \quad (24)$$

and we obtain that

$$\begin{aligned}
\delta S &= \int d^4x [\delta a_i {}_0D_t^\alpha \eta^i + a_i \delta({}_0D_t^\alpha \eta^i) - \delta H] \\
0 &= \int d^4x \left[ \frac{\partial a_i}{\partial \eta^j} \delta \eta^j {}_0D_t^\alpha \eta^i + a_i {}_0D_t^\alpha \delta \eta^i - \delta H \right] \\
0 &= \int d^4x \left[ \frac{\partial a_i}{\partial \eta^j} \delta \eta^j {}_0D_t^\alpha \eta^i - {}_0D_t^\alpha a_i \delta \eta^i - \frac{\partial H}{\partial \eta^j} \delta \eta^j \right] \\
0 &= \int d^4x \left[ \frac{\partial a_i}{\partial \eta^j} \delta \eta^j {}_0D_t^\alpha \eta^i - \frac{\partial a_i}{\partial \eta^j} {}_0D_t^\alpha \eta^j \delta \eta^i - \frac{\partial H}{\partial \eta^j} \delta \eta^j \right] \\
0 &= \int d^4x \left[ \left( \frac{\partial a_i}{\partial \eta^j} - \frac{\partial a_j}{\partial \eta^i} \right) {}_0D_t^\alpha \eta^i - \frac{\partial H}{\partial \eta^j} \right] \delta \eta^j,
\end{aligned} \tag{25}$$

and so, if the  $\delta \eta^j$  are independent, we see that

$${}_0D_t^\alpha \eta^i = [f_{ij}^{(0)}]^{-1} \frac{\partial H}{\partial \eta^j}. \tag{26}$$

and

$$f_{ij} = \frac{\partial a_i}{\partial \eta^j} - \frac{\partial a_j}{\partial \eta^i}, \tag{27}$$

is the usual symplectic matrix as considered before.

A very important point to be stressed here is that, despite our last result could appear usual, it is really not, for the reason that we could get it in a fractional context only because we employed the chain rule in the MRL context. The most recent literature has presented some interesting results, but always building up theories and models through Riemann-Liouville or Caputo definitions. In these cases, however, in spite the success of such approaches, the chain rule is always avoided.

## V. COARSE-GRAINING EMBEDDING

### A. Dirac-Bergmann Algorithm in the Coarse Graining

It has been often realized that some sort of coarse-graining is necessary in order that typically quantum features of a system (with finite number of degrees of freedom) do not dominate over other contributions. The coarse-graining enters differently in diverse theories of quantum to classical relation, and is not always equally strongly emphasized. In the theories of decoherence [32], the emphasis is on the influence of the environment, but the description of the environment must be coarse-grained to fulfill the desired decoherence effects. In this way, we believe that is important to adapt some quantization procedures to this reality. So, we shall now consider a more generalized approach to the symplectic algorithm. We focus our attention on understanding the behavior of Dirac bracket in a coarse-graining context and we will again use the MRL prescriptions for that purpose. So, let us start by considering a coarse-grained configuration space,

$$\Xi^\alpha = \{\mathbf{q}_i, \mathbf{p}_i, \mathbf{t}\} \tag{28}$$

we have not adopted the label  $\{q_i^\alpha, p_i^\alpha, t\}$  in the coordinates to have a more simplified notation and avoid subscript saturation. We now understand that bold face coordinates are embedded in a fractional space-time context. Let us consider an action within that  $\alpha$ -phase space

$$S = \int_t^{t'} [\mathbf{p}_i {}_0D_t^\alpha \mathbf{q}_i - H(\mathbf{p}, \mathbf{q})] (dt)^\alpha \tag{29}$$

where the summation convention for repeated indices is considered once more. The variation of (29) implies

$$\begin{aligned}
\delta S &= \int_t^{t'} [\delta^\alpha \mathbf{p}_i {}_0D_t^\alpha \mathbf{q}_i + \mathbf{p}_i {}_0D_t^\alpha \delta^\alpha \mathbf{q}_i + \\
&\quad - \frac{\partial^\alpha H}{\partial \mathbf{p}_i^\alpha} (\delta p_i)^\alpha - \frac{\partial^\alpha H}{\partial \mathbf{q}_i^\alpha} (\delta q_i)^\alpha] (dt)^\alpha = 0
\end{aligned} \tag{30}$$

Based on the MRL formulation [29, 30], we can employ the following approximation

$$(\delta u_i)^\alpha \approx (\alpha!)^{-1} \delta^\alpha \mathbf{u}_i, \tag{31}$$

and, after some little algebra, we can obtain the fractional extension for the Hamilton-Jacobi equations of motion within the MRL context:

$$\begin{aligned}
{}_0D_t^\alpha \mathbf{q}_i &\approx (\alpha!)^{-1} \frac{\partial^\alpha H}{\partial \mathbf{p}_i^\alpha} \\
{}_0D_t^\alpha \mathbf{p}_i &\approx -(\alpha!)^{-1} \frac{\partial^\alpha H}{\partial \mathbf{q}_i^\alpha}.
\end{aligned} \tag{32}$$

Considering now dynamical variables embedded in a coarse-grained spacetime,  $\Theta = \Theta(\mathbf{q}_i, \mathbf{p}_i, \mathbf{t})$  for instance, it is reasonable to expand them by means of a fractional Taylor's series [29]

$$\begin{aligned}
{}_0D_t^\alpha \Theta &= \frac{\partial^\alpha \Theta}{\partial \mathbf{q}_i^\alpha} (Dq_i)^\alpha + \frac{\partial^\alpha \Theta}{\partial \mathbf{p}_i^\alpha} (Dp_i)^\alpha + \frac{\partial^\alpha \Theta}{\partial t^\alpha} (Dt)^\alpha \\
{}_0D_t^\alpha \Theta &\approx \frac{1}{\alpha!} \left( \frac{\partial^\alpha \Theta}{\partial \mathbf{q}_i^\alpha} ({}_0D_t^\alpha \mathbf{q}_i) + \frac{\partial^\alpha \Theta}{\partial \mathbf{p}_i^\alpha} ({}_0D_t^\alpha \mathbf{p}_i) \right) + \frac{1}{\alpha!} \frac{\partial^\alpha \Theta}{\partial t^\alpha} ({}_0D_t^\alpha \mathbf{t}).
\end{aligned} \tag{33}$$

Using the Hamilton-Jacobi equations (32), we can adapt an approximated fractional version (MRL sense) for the Poisson bracket

$$\{U, V\}_\alpha = \left( \frac{1}{\alpha!} \right)^2 \left( \frac{\partial^\alpha U}{\partial \mathbf{q}_i^\alpha} \frac{\partial^\alpha V}{\partial \mathbf{p}_i^\alpha} - \frac{\partial^\alpha U}{\partial \mathbf{p}_i^\alpha} \frac{\partial^\alpha V}{\partial \mathbf{q}_i^\alpha} \right), \tag{34}$$

where  $U$  and  $V$  are two dynamical variables defined in phase space. Though we have presented an approximated treatment, our approach seems reasonable to treat the quantization of nonlinear systems. It is clear that there exists an extensive literature describing techniques for



relating Lagrangian physical systems in the linear regime, however it was only over the past 15 years or so that there has been a general awareness of the possibility that irregular-looking fluctuations may be caused by deterministic chaotic dynamics [33]. In this sense, we believe that new perspectives for quantize these systems must be investigated and the MRL symplectic approach can be useful to search for new results.

In a more general sense, it is possible to obtain the right canonical structure for some fractional constrained system. Roughly speaking, the natural extension of the Dirac bracket obeys the same algebraic protocol, but starts off with some Hamiltonian,

$$\tilde{H} = H + \lambda_m \phi_m, \quad (35)$$

where  $H$  is the canonical Hamiltonian,  $\lambda_m$  are Lagrange multipliers and  $\phi_m$  are coarse-grained constraints. More precisely,

$$\phi = \phi(\mathbf{q}_i, \mathbf{p}_i). \quad (36)$$

Following the usual steps, we can obtain Hamilton-Jacobi equations for constrained systems in a coarse-grained scenario,

$$\begin{aligned} {}_0D_t^\alpha \mathbf{q}_i &\approx (\alpha!)^{-1} \left( \frac{\partial^\alpha H}{\partial \mathbf{p}_i^\alpha} + \lambda_m \frac{\partial^\alpha \phi_m}{\partial \mathbf{p}_i^\alpha} \right) \\ {}_0D_t^\alpha \mathbf{p}_i &\approx -(\alpha!)^{-1} \left( \frac{\partial^\alpha H}{\partial \mathbf{q}_i^\alpha} + \lambda_m \frac{\partial^\alpha \phi_m}{\partial \mathbf{q}_i^\alpha} \right), \end{aligned} \quad (37)$$

and, consequently, we can write the right form for the coarse-grained Dirac bracket

$$\{U, V\}_\alpha^* = \{U, V\}_\alpha - \{U, \phi_i\}_\alpha [C_{ij}^\alpha]^{-1} \{\phi_j, V\}_\alpha, \quad (38)$$

where  $C_{ij}^\alpha$  is the matrix of constraints, usually defined on Dirac algorithm, in our case we will define it subsequently in (44).

### B. Linking Symplectic and Dirac-Bergmann algorithms in the Coarse Grained Scenario

Our goal now will be to relate both algorithms through the MRL<sup>1</sup> prescription. So, let us consider first an action embedded in some coarse-grained space-time,

$$S = \int (dx)^\alpha [a_i(\boldsymbol{\eta}) {}_0D_t^\alpha \boldsymbol{\eta}_i(x) - V(\boldsymbol{\eta})], \quad k = 1, 2, \dots, 2n \quad (39)$$

<sup>1</sup> once more it is important to emphasize here that  $\alpha$  suffix is the fractional degree of our coarse-graining physical space; no confusion must be done with Lorentz covariant suffix or other ones.

Again, we are dealing with a more simplified notation, where the field variables do not have any fractional tag; therefore, we adopt the same prescription and use bold face variables, so that the action above gives rise to the Euler-Lagrange equations to describe our fractional scenario,

$$f_{ij}^\alpha {}_0D_t^\alpha \boldsymbol{\eta}^i = \frac{\partial^\alpha H}{\partial \boldsymbol{\eta}^{\alpha j}}, \quad (40)$$

where the  $f_{ij}^\alpha$  is the symplectic matrix

$$f_{ij}^\alpha = \frac{\partial^\alpha a_i}{\partial \boldsymbol{\eta}^{\alpha j}} - \frac{\partial^\alpha a_j}{\partial \boldsymbol{\eta}^{\alpha i}} \quad (41)$$

however, from the Dirac point of view, we can think that the conjugate momentum here is given by

$$p_i = \frac{\partial^\alpha L}{\partial ({}_0D_t^\alpha \boldsymbol{\eta}_i)^\alpha} = a_i, \quad (42)$$

the so-called primary constraint is then

$$\Omega_i = p_i - a_i \approx 0. \quad (43)$$

The Dirac bracket structure defines a matrix of constraints  $C_{ij}$ ; in our case, we have

$$C_{ij}^\alpha = \{\Omega_i, \Omega_j\}_\alpha = (\alpha!)^{-2} \left( \frac{\partial^\alpha a_i}{\partial \boldsymbol{\eta}^{\alpha j}} - \frac{\partial^\alpha a_j}{\partial \boldsymbol{\eta}^{\alpha i}} \right), \quad (44)$$

or, in a more concise way,

$$C_{ij}^\alpha = (\alpha!)^{-2} f_{ij}^\alpha. \quad (45)$$

Considering (45), we are ready to establish the right connection between both approaches

$$\begin{aligned} \{\eta_i, \eta_j\}_\alpha^* &= \{\eta_i, \eta_j\}_\alpha - \{\eta_i, \phi_r\}_\alpha (\alpha!)^2 [f_{rs}^\alpha]^{-1} \{\phi_s, \eta_j\}_\alpha \\ \{\eta_i, \eta_j\}_\alpha^* &= (\alpha!)^{-2} \left[ (\delta_{ir}) [f_{rs}^\alpha]^{-1} (\delta_{js}) \right] \\ \{\eta_i, \eta_j\}_\alpha^* &= (\alpha!)^{-2} [f_{ij}^\alpha]^{-1}. \end{aligned} \quad (46)$$

Of course that the same steps considered in the Section II can be repeated here. We obtained a new corrected symplectic matrix and, reconsidering our chain of reasoning, it becomes apparent that we extended a well-known quantization method to study the response of a coarse-grained-based system when the transition from classical to quantum is considered.

## VI. EXAMPLE AND APPLICATION

### A. A Simple Model

Let us now present a useful example to apply our result. In spite it is a well-tested result, the usual case was considered in [34]. It is well-known that charged particles moving on a plane with velocity  $\dot{r}$ , subject to an external magnetic field, modulus  $B$ , perpendicular to the plane,

has the following equation of motion, governed by the Lorentz force:

$$\ddot{r}^j = \frac{eB}{m} \epsilon^{ij} \dot{r}_i, \quad (47)$$

Our goal here is to implement the fractional approach. We define a fractional temporal operator. We must observe that this operator will impose a new time unity ( $T^\alpha$ ), so, for that reason, we will re-scale all units of length with the help of label ( $L^\alpha$ ). Physically, we could think of some heterogeneous planar system with different layers and different densities. Today, graphene monolayers show that genuinely planar systems are physically realizable.

For that purpose, the natural fractional extension is given by the following Lagrangian,

$$L = k^\alpha \left[ \frac{1}{2} m \left( {}_0D_t^\alpha r_i^\alpha \right)^2 + e A^{\alpha i} {}_0D_t^\alpha r_i^\alpha - V(r^\alpha) \right] \quad (48)$$

We know that the properties of fractional calculus (derivatives and integrals) are not the same as in the usual calculus. Therefore, we believe that with this approach, we can access some new perspectives, for instance, in complex systems where memory effects are very useful to describe suitably their dynamics. The equation of motion for the Lagrangian (48) is <sup>2</sup>

$${}_0D_t^\alpha {}_0D_t^\alpha r^{\alpha i} = \frac{2e}{m} {}_0D_t^\alpha A^{\alpha i}, \quad (49)$$

and, by choosing the Landau symmetric gauge,

$$A^{i\alpha} = \frac{1}{2} B^\alpha \epsilon^{ij} r_j^\alpha \quad (50)$$

leaves

$${}_0D_t^\alpha {}_0D_t^\alpha r^{j\alpha} = \frac{eB^\alpha}{m} \epsilon^{ij} {}_0D_t^\alpha r_i^\alpha. \quad (51)$$

If we consider the MRL differential relation  $D^\alpha f \approx \Gamma(1+\alpha)Df$  once more, the differential equation (51) can be cast under the form

$$\ddot{r}^j \approx \frac{eB}{m\Gamma(1+\alpha)} \epsilon^{ij} \dot{r}_i \quad (52)$$

if we look upon the Hall effect in the usual context, we know that the quantized theory yields the Landau levels when external forces are not involved. So, in that regime, and performing the change for the complex notation, considering  $z = x + iy$ , we rewrite the expression (52) as

$$\ddot{z} \approx \frac{i\omega}{\Gamma(1+\alpha)} \dot{z}, \quad (53)$$

giving rise to us the following solution,

$$z \approx z_0 + d \exp\left(\frac{i\omega}{\Gamma(1+\alpha)} t\right), \quad (54)$$

where  $z_0$  is usually considered arbitrary and called, guiding center, the constant  $d$  is the radius of cyclotron.

## B. Estimating the Fractionality

We understand that there is an important and interesting discussion under consideration. It is possible to notice that the cyclotron frequency gets a fractional correction  $\omega^\alpha = \omega/\Gamma(1+\alpha)$ , that our approach allows us to compute. In this context, we intend to investigate the two possible limits for  $\alpha$ , and we shall base our analysis by requiring that the ratio below is close to the relative error in the cyclotron frequency,

$$\frac{\omega_\alpha - \omega}{\omega} \approx 10^{-8} \quad (55)$$

In the first limit, we consider ( $\alpha \rightarrow 0$ ); then, we can take the gamma function after its usual expansion

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) = 1 - \alpha\gamma, \quad (56)$$

where  $\gamma = 0.57721$  is the Euler-Mascheroni constant. Consequently, using the geometric expansion, the cyclotron frequency can be re-written now as

$$\omega^\alpha = \frac{\omega}{1 - \alpha\gamma} = \omega(1 + \alpha\gamma). \quad (57)$$

using the relation (55), we can write that

$$\alpha \approx 10^{-8}. \quad (58)$$

Curiously,  $\alpha$  displays an interesting correlation with the noncommutative parameter  $\theta$  [35, 36]; indeed the order of magnitude for the magnetic field is the same as the one in the Hall materials.

Now, let us consider the limit when ( $\alpha \rightarrow 1^-$ ). We shall adopt a similar, but not quite the same, approach as the one in [37]. Let us apply once more the ratio (55)

$$\begin{aligned} \frac{\omega_\alpha - \omega}{\omega} &\approx 10^{-8} \\ \frac{1}{\Gamma(1+\alpha)} - 1 &\approx 10^{-8} \\ \alpha &\approx 0.9999999763473 \end{aligned} \quad (59)$$

after considering the well-known experimental values and error bars of the Hall effect metrology [38]. It is important to remark that our last result points to the so-called low-level fractionality limit.

<sup>2</sup> it is important to remark that  $D^{2\alpha}f \neq D^\alpha D^\alpha f$  and  $k^\alpha$  is a constant to adjust the Lagrangian unit for Joule. It will remains equal 1 for  $D^\alpha \approx \Gamma(1+\alpha)D$

### C. Quantization

Let us now consider the Lagrangian (48) in the limit of intense  $B$  and ( $m \rightarrow 0$ ) as taken in the same gauge

$$L = k^\alpha \left[ \frac{1}{2} e B^\alpha \epsilon^{ij} r_j^\alpha {}_0D_t^\alpha r_i^\alpha - V(r^\alpha) \right] \quad (60)$$

it is clear that we are dealing with a Lagrangian like (23), where the canonical pair is given here by

$$(eB\Gamma(1+\alpha)r_i, r_j), \quad (61)$$

allowing us to make the right identification  $H = V(r)$ . Applying the canonical formalism, by means of symplectic algorithm, we can observe that our set of symplectic variables are  $\eta = \{r_1, r_2\}$  or  $\{x, y\}$  and the kinetic parts read

$$a_r^i = \frac{1}{2} e B \Gamma(1+\alpha) \epsilon^{ij} r_j, \quad (62)$$

Following (27), we obtain by direct calculation

$$f^{ij} = eB\Gamma(1+\alpha)\epsilon^{ij}, \quad (63)$$

leading us to the Dirac bracket

$$(f^{ij})^{-1} = \{r_i, r_j\}^* = \frac{\epsilon_{ij}}{eB\Gamma(1+\alpha)}. \quad (64)$$

The last equation can be re-written if we again consider the limit ( $\alpha \rightarrow 0$ ). The Dirac bracket is

$$\{r_i, r_j\}^* = \frac{\epsilon_{ij}}{eB}(1+\alpha\gamma), \quad (65)$$

and the commutator

$$[r_i, r_j] = i\hbar \frac{\epsilon_{ij}}{eB}(1+\alpha\gamma). \quad (66)$$

The form of the commutator can be understood as a contribution from the fractionality, even if it is small. The usual and expected noncommutative structure, given by

$$[r_i, r_j] = i\theta_{ij} \quad (67)$$

gets a fractional correction in the limit ( $\alpha \rightarrow 0$ ). This interesting new perspective may be putting in evidence a reasonable path for helping future investigations on high energy physics at Planck scale. Another important point is that, in this limit,  $\theta$  and  $\alpha$  seems to have the same order of magnitude.

The noncommutativity of the space coordinates should not after all be surprising in a coarse-grained framework. There is always an intrinsic uncertainty in the position coordinates, so that a noncommutative character is very reasonable.

### VII. CONCLUSIONS AND PERSPECTIVES

In summary, our main motivation is to develop an approach based on the fractional variational calculus. More precisely, the modified Riemann-Liouville fractional derivative, to address coarse-grained constrained systems, since FC shows itself a powerful tool to treat this class of systems. Our effort is also to present a fractional symplectic algorithm for systems with a higher extent of complexity. Consequently, we have shown that it is possible to discuss quantization in this scenario and clarify its correspondence by means of the MRL approach. We have proposed three kinds of fractional formulations for the symplectic algorithm.

The first formulation considers a Lagrangian model like  $L = \eta f^{(0)} {}_0D_t^\alpha \eta - H(\eta)$  within the MRL approach. We have obtained the Hamilton-Jacobi equations deformed by the fractional contribution in the symplectic scenario along with the symplectic matrix. The corresponding Dirac brackets have also received the same kind of modification.

However, when we enlarge our class of applications by considering other field theories, this first approach does not seem to be the more general way to treat them. Therefore, we have convinced ourselves on the necessity to generalize the Lagrangian by considering  $L = a_i(\eta) {}_0D_t^\alpha \eta - V(\eta)$ . The constraints were defined in the same way and the consequence was the extension of the symplectic matrix in a fractional framework. We have obtained the final form for the fractional equations of motion and the symplectic matrix associated to the Dirac brackets, which now exhibit an additional term  $\Gamma(1+\alpha)$ , due to the fractional contribution imposed by the fractional formalism. An important point is that we can recover the usual brackets whenever  $\alpha \rightarrow 1$ .

Our third formulation emphasizes the action in a coarse-grained scenario. We suppose to be working with some configuration space where the coordinates are embedded in a coarse-grained space and analyze its unfolds by means of the MRL approach. Our first step considers an extension of the Dirac-Bergmann algorithm and we present the corrected expressions for the Hamilton-Jacobi equations when constraints are present. Subsequently, we connect the Dirac-Bergmann algorithm with the symplectic quantization scheme and obtain an interesting new relationship between Dirac and Symplectic matrices, now corrected by an  $\alpha^{-2}$  factor.

Evidently, other different definitions for the fractional derivative could be used with the same purpose. For example, the generalized Euler formula, Abel or Fourier integral representation, Riemann-Liouville, Caputo, Sonin, Letnikov, Laurent, Nekrasov and Nishimoto representations, for example. But, we have to recall once more that, in all these approaches, the chain rule becomes rather messy, yielding very lengthy calculations. For this reason, perhaps, FC has found some resistance to become more familiar in areas of Physics such as gravitation and field theory.



However, more recently, some new approaches have been presented that could yield similar results. For instance, [39, 40] or the approaches with Hausdorff derivative, also referred to as fractal derivative [41], can be applied to power-law phenomena and the recently developed  $\alpha$ -derivative [42].

We still presented an example where a fractional Lagrangian extension was considered that describes charged particles in a magnetic field. In this case, the time derivative has been changed by the fractional derivative, and the fractional symplectic algorithm was then implemented. We have obtained an  $\alpha$ -corrected expression for the Dirac brackets, and we have analyzed its approximated form whenever  $\alpha \rightarrow 0$ . A very close analogy between  $\alpha$  and the noncommutative parameter  $\theta$  has been found out.

It is instructive to notice that the granularity of the system can be represented by the fractional parameter  $\alpha$ . So, we can observe two different situations. The first one, when  $\alpha \rightarrow 0$ , we can think that the medium granularity is gross, so the system can be understood as rough or sparse. In this limit, comparing the processes involved, the dynamics of the system can be understood as very slow, namely, the relaxations processes are slow, since the derivatives in the dynamical equations tend to zero.

In our second limit, when  $\alpha \rightarrow 1^-$ , we could talk about tiny or fine granularity, represented by the so called low-level fractionality. In this case, the system is almost continuous and not sparse. The interactions between parts of the systems and the environment are more frequent and the relaxations are faster. The dynamics can be better understood and discussed in terms of complexity of these interactions. The anomalous behavior of the system can be connected with the non-locality of the interactions. Taking this point of view, small discrepancies on the fundamental constants can be expected.

In the MRL approach, eq. (15), we always work with  $\alpha$  such that  $0 < \alpha < 1$ . We could consider the extension of  $\alpha$  to other ranges; specifically, if  $\alpha < 0$ , we are in a regime

of a different dynamics, whose equations are expressed in terms of integrals rather than derivatives. This highlights some singularities of our results given in Section VI and relate the question with the Landau problem on the non-commutative plane [43, 44].

Another point we could consider in a further work regards the procedures of reduction and quantization for constrained systems. We could reassess the work [45] in a scenario with FC. The discussion on the prescriptions of "first reduce and then quantize" or "first quantize and then reduce" becomes of interest in connection with FC.

We have also come to the conclusion that a fractional model can provide us with a memory effect in the convolution integrals and leads to some differential equations which could open up different physical situations, such as viscoelasticity and more abstract scenarios such as mapping using tensorial fields.

Besides the applications presented here, we strongly believe that quantization in a fractional context is a widely open area that may raise a tremendous deal of interest. As a viable and almost immediate extension, we could think about the coarse-grained formulation of a fractional Dirac equation without taking it as the  $(\sqrt{\square})$ . We shall be reporting our studies on that in a forthcoming work.

We do not know yet the whole bunch of problems that can be handled using this approach. Several aspects of gravitation, condensed matter systems and field theory seem to be ready to be approached in terms of the FC and its tools.

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